Instructions: (1) All questions carry equal marks.
(2) Use of calculator and statistical tables is permissible.

1 (a) Define a probability space and show that probability measure $P$ defined on the space is countably subadditive.
(b) Define a distribution function of random variable and show that set of discontinuity points of a distribution function is atmost countable.
(c) State the decomposition theorem for the distribution functions.
A distribution function is given by
$$F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{x^2}{3} & \text{if } 0 \leq x < 1 \\
\frac{1}{3} + \left(\frac{x-1}{2}\right)^2 & \text{if } 1 \leq x < 2 \\
1 & \text{if } x \geq 2. 
\end{cases}$$
Obtain the decomposition of $F$ into its discrete and continuous parts.

OR

1 (a) State and prove Markov’s inequality. Derive Chebyshev’s inequality from Markov’s inequality.
(b) If $E(X_1) = E(X_2) = 0$, $V(X_1) = V(X_2) = 1$ and $\text{COV}(X_1, X_2) = \rho$ then show that $E\left[\max(X_1^2, X_2^2)\right] \leq 1 + \sqrt{1 - \rho^2}$. Hence derive the BERGE inequality for correlated random variables.
(c) Show that convergence almost surely implies convergence in probability.

2 (a) Define characteristic function of random variable. State and prove inversion theorem on characteristic function.
(b) State Borel Cantelli Lemma. If \( \{A_n\} \) is a sequence of events such that \( P(A_n) = \frac{1}{2} P(A_{n-1}) \) and \( A_1 \) is a certain event then find \( P(\lim \sup A_n) \).

(c) State and prove Helly-Bray theorem.

OR

2 (a) \( \{X_n\} \) be a sequence of i.i.d. random variables with \( E(X_n) = \mu < \infty \). Then show that \( \bar{X}_n \xrightarrow{p} \mu \).

(b) State and prove Liapounov's form of central limit theorem.

(c) If \( \{X_n\} \) is a sequence of independent random variables with the distributions: \( P[X_n = \pm n^\lambda] = \frac{1}{2} \) where \( \lambda \) is constant. Examine whether the central limit theorem holds for any \( \lambda \).

3 (a) Define the terms:
(i) Markov Chain
(ii) Absorbing State
(iii) Stationary distribution.

(b) \( P(X_0 = i) = \frac{1}{3} \) for \( i = 1, 2, 3 \) and

\[
P = \begin{bmatrix}
0 & \frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{2}{3} \\
\frac{2}{3} & \frac{1}{3} & 0
\end{bmatrix},
\]
then compute \( P(X_2 = i) \) for \( i = 1, 2, 3 \).

(c) Stating the postulates of a pure birth process, obtain the difference-differential equations governing,

\( P_n(t) = P(x(t) = n | X(0) = i) \)

OR

3 (a) Define a contagious distribution and obtain its probability generating function. Hence or otherwise derive the probability function of Poisson-Pascal distribution.

(b) Let \( X_1, X_2, ..., X_N \) be independent random variables with common distribution \( F \) and let \( N \) be a random variable independent of \( X_j \). Define \( Y = \sum_{i=1}^{N} X_i \) and let \( \phi_t(t) \) be the characteristics function (ch.f) of \( Y \). \( \phi_1(t) \) and \( \phi_2(t) \) are the
ch.fs of random variables $N$ and $X$ respectively then show that $\phi_Y(t) = \phi_1[-1 \log(\phi_2(t))]$.

(c) Explain the method to obtain ML estimates of the parameters of Neyman type-A distribution.

4 (a) Define Poisson-Binomial distribution and derive its $r^{th}$ factorial cumulant. Hence obtain moment estimators of the parameters of the distribution.

(b) Suppose the number of automobile accidents in a certain area in a week is a random variable with mean $\mu_1$ and variance $\sigma_1^2$. The number of persons in sured in each accident is independent from accident to accident and has mean $\mu_2$ and variance $\sigma_2^2$. Let $W$ denote the number of persons injured in automobile accidents in the area in a week. Find $E(W)$ and $V(W)$.

(c) Define multinomial distribution $M_k(n, p_1, p_2, \ldots, p_k)$ for the distribution show that

$$
\rho_{1234\ldots m} = -\left(\frac{p_1 p_2}{(1-p_1-p_3-\ldots-p_m)(1-p_2-p_3-\ldots-p_m)}\right)^{1/2}
$$

where $m < k$.

OR

4 (a) Define multivariate normal distribution. Let $x \sim N_n(\mu, \Sigma)$ then show that $y = Lx \sim N_p(L\mu, L\Sigma L')$ where $L$ is a known matrix of order $n \times p$. Hence show that $y_1 = L_1 x$ and $y_2 = L_2 x$ are independently distributed as normal iff $L_1 \Sigma L_2 = 0$.

(b) Let $x \sim N_p(\mu, \Sigma)$. Derive the distribution of $X'AX$, where $A$ is positive definite real symmetric matrix. Hence show that $E(X'AX) = tr(A\Sigma) + \mu' A \mu$.

(c) Let $X_n = \maxi (X_1, X_2, \ldots, X_n)$. Show that

$$E(X_{(n)}) = E(X_{(n-1)}) + \int_{0}^{\infty} F^{n-1}(t)[1-F(x)]dx, \quad n = 2, 3, \ldots.$$ Also find $E(X_{(n)})$ if $X_1$'s have the common distribution function $F(x) = 1 - e^{-\beta x}$, $x \geq 0$, $\beta > 0$.

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(a) Let $X_1, X_2, ..., X_n$ be iid $N(0,1)$ random variables. Define $Y = \sum_{i=1}^{n} (X_i + \lambda)^2$. Obtain the sampling distribution of $Y$. Also find the $r^{th}$ cumulant of $Y$. What will be the distribution of $Y$ when $\lambda = 0$?

(b) Define non-central $F$ and doubly non-central $F$ variates. Derive the distribution of non-central $F$ variate and find its mean.

**OR**

(a) If $X$ follows $N(\mu, 1)$ distribution and $Y$ is an independent chi-square variate with $n$ degrees of freedom then obtain the distribution of $\frac{\sqrt{n}X}{\sqrt{Y}}$. Give some applications of the distribution.

(b) Prove re-productive property of noncentral chi-square distribution. If $X$ and $Y$ are two independent random variables such that $X$ is chi-square variate with $r$ df and $Y$ is noncentral chi-square with 1 df and $\lambda$ non-centrality parameter then obtain the distribution of $X+Y$. Find its mean and variance.